Multi-vortex solution in the Sutherland model

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Abstract. We consider the large-N Sutherland model in the Hamiltonian collective-field approach based on the $1/N$ expansion. The Bogomol'nyi limit appears and the corresponding solutions are given by static-soliton configurations. They exist only for $\lambda < 1$, i.e. for the negative coupling constant of the Sutherland interaction. We determine their creation energies and show that they are unaffected by higher-order corrections.

1. Introduction

Classical and quantum solitons in the Calogero–Sutherland model have recently been intensively studied [1–3]. The underlying motivation is that these extended objects may presumably play an important role in a deeper understanding of quasiparticle and quasihole physics of exact solution [4]. The collective theory of [3] offers a field-theoretic framework for describing semi-classical soliton configurations in the large-N limit. We have recently shown that in the Calogero model there exist soliton-type finite-energy solutions that can be obtained by solving a first-order integro-differential equation of Bogomol'nyi type.

In this paper we are primarily concerned with static solitons in the collective-field formulation of the Sutherland model. These solitons can also be reached by the Bogomol'nyi saturation. Among them there are some new, periodic multi-vortex solutions which have not been discussed so far. Their existence stems from the fact that the Sutherland model is defined on the compact support (circle) and therefore must satisfy periodic boundary conditions.

2. The collective-field Hamiltonian

The Sutherland system has a Hamiltonian describing spinless particles confined to a ring and interacting through a $1/r^2$ pairwise potential [5]:

$$H = \frac{4\pi^2}{L^2} \left(-\frac{1}{2} \sum_{i=1}^{N} \frac{d^2}{d\varphi_i^2} + \frac{g}{4} \sum_{i>j}^{N} \frac{1}{\sin^2\left((\varphi_i - \varphi_j)/2\right)} \right) \tag{1}$$

where L is the length of the ring and N is the number of particles. We use units in which $\hbar = m = 1$, with m being the mass of the particles. Here φ_i is the angular coordinate of the ith particle. The dimensionless coupling constant g determines the strength of the

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Sutherland pair coupling and is related to the statistical parameter λ of the exclusion statistics [6] by the relation

$$g = \lambda(\lambda - 1). \tag{2}$$

For special values of λ, i.e. $\lambda = 0$, we have free bosons, and for $\lambda = 1$ and 2 the model is related to a system of free fermions and to the Haldane–Shastry spin chain [7]. Because of the singularity of the Hamiltonian for $\varphi_i = \varphi_j$, the wavefunction ought to have a prefactor that vanishes for coinciding particles:

$$\psi = \Delta^\lambda \phi \qquad \Delta = \prod_{i<j} \sin\left(\frac{\varphi_i - \varphi_j}{2}\right). \tag{3}$$

With this factorization, we obtain a new Hamiltonian that acts on the residual, completely symmetric wavefunction ϕ:

$$H = -\frac{1}{2}\sum_{i=1}^{N} \frac{\mathrm{d}^2}{\mathrm{d}\varphi_i^2} - \frac{\lambda}{2}\sum_{i=1}^{N}\left(\sum_{j\neq i}^{N} \cot\frac{\varphi_i - \varphi_j}{2}\right)\frac{\mathrm{d}}{\mathrm{d}\varphi_i} + \frac{\lambda^2}{24}N(N^2 - 1). \tag{4}$$

The last constant term emerges from the trigonometric identity

$$\sum_{i\neq j\neq k} \cot\frac{\varphi_i - \varphi_j}{2} \cot\frac{\varphi_i - \varphi_k}{2} = -\frac{1}{3}N(N - 1)(N - 2). \tag{5}$$

The non-trivial part of the Sutherland Hamiltonian is now suitable for transformation into a collective-field representation. For the collective field we take the permutation symmetric function

$$\rho(\varphi) = \sum_{i=1}^{N} \delta(\varphi - \varphi_i) \tag{6}$$

obeying the normalization condition

$$\int_0^{2\pi} \mathrm{d}\varphi \, \rho(\varphi) = N. \tag{7}$$

Next, we reformulate the differential operators in the Hamiltonian (4) in terms of a functional differentiation with respect to the collective field $\rho(\varphi)$. For $\lambda = 1$, we obtain the result derived previously by two of us [8] that shows the equivalence of the collective-field version of the Sutherland model with the one-plaquette restriction of Kogut–Susskind $U(N)$ gluodynamics [9]. For general λ, we proceed in a similar way. Using the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}\varphi_i} = \int \mathrm{d}\varphi \, \frac{\partial\rho(\varphi)}{\partial\varphi_i} \frac{\delta}{\delta\rho(\varphi)} \tag{8}$$

and by rescaling the wavefunction

$$\phi(\varphi_1, \ldots, \varphi_N) = J^{1/2}\Phi(\rho) \tag{9}$$

after some calculation we find the Hermitian collective-field Hamiltonian

$$H = \frac{1}{2}\int \mathrm{d}\varphi \, \rho(\varphi)(\partial_\varphi\pi(\varphi))^2 + \frac{1}{8}\int \mathrm{d}\varphi \, \rho(\varphi)\left(\partial_\varphi \frac{\delta \ln J}{\delta\rho(\varphi)}\right)^2$$

$$-\frac{\lambda - 1}{4}\int \mathrm{d}\varphi \, \partial_\varphi^2\delta(\varphi - \varphi')|_{\varphi=\varphi'} - \frac{\lambda}{4}\int \mathrm{d}\varphi \, \rho(\varphi)\partial_\varphi \cot\frac{\varphi - \varphi'}{2}\bigg|_{\varphi=\varphi'}$$

$$+\frac{\lambda^2}{24}N(N^2 - 1). \tag{10}$$

Here $\pi(\varphi)$ is the canonical conjugate of the field $\rho(\varphi)$:

$$[\partial_\varphi \pi(\varphi), \rho(\varphi')] = -i\partial_\varphi \delta(\varphi - \varphi'). \tag{11}$$

The Jacobian J is determined from the hermiticity condition

$$\partial_\varphi \left(\rho(\varphi)\partial_\varphi \frac{\delta \ln J}{\delta \rho(\varphi)} \right) = (\lambda - 1)\partial_\varphi^2 \rho(\varphi) + \lambda \partial_\varphi \left(\rho(\varphi) \fint d\varphi' \ \cot \frac{\varphi - \varphi'}{2} \rho(\varphi') \right) \tag{12}$$

and reads

$$J = \exp\left[(\lambda - 1) \int d\varphi \ \rho(\varphi) \ln \rho(\varphi) + \frac{\lambda}{2} \int d\varphi \ d\varphi' \ \rho(\varphi) \ln \sin^2 \frac{\varphi - \varphi'}{2} \rho(\varphi') \right]. \tag{13}$$

The two singular terms in the Hamiltonian (10) do not contribute to leading order in N. They should be cancelled by the infinite zero-point fluctuations of the collective field $\rho(\varphi)$. This will be discussed in detail in section 4.

3. The Bogomol'nyi limit

To find the ground-state energy and the corresponding collective motion in the large-N limit, we should minimize the energy functional with respect to $\pi(\varphi)$ and $\rho(\varphi)$. However, in our case, owing to the special features of the model, there is a much more efficient method for minimization. The leading term of the collective-field Hamiltonian in the $1/N$ expansion is given by the effective potential

$$V_{\text{eff}} = \frac{1}{8} \int d\varphi \ \rho(\varphi) \left(\partial_\varphi \frac{\delta \ln J}{\delta \rho(\varphi)} \right)^2$$

$$= \frac{1}{2} \int d\varphi \ \rho(\varphi) \left(\frac{\lambda - 1}{2} \frac{\partial_\varphi \rho(\varphi)}{\rho(\varphi)} + \frac{\lambda}{2} \fint d\varphi' \ \cot \frac{\varphi - \varphi'}{2} \rho(\varphi') \right)^2. \tag{14}$$

Owing to the positive definiteness of the effective potential (14), the Bogomol'nyi limit appears. The Bogomol'nyi bound is saturated by the positive, normalizable solution $\rho_0(\varphi)$ of the equation

$$\frac{\lambda - 1}{2} \frac{\partial_\varphi \rho(\varphi)}{\rho(\varphi)} + \frac{\lambda}{2} \fint d\varphi' \ \cot \frac{\varphi - \varphi'}{2} \rho(\varphi') = 0 \tag{15}$$

with the ground-state energy equal to

$$E_0 = \frac{1}{24}\lambda^2 N(N^2 - 1) \tag{16}$$

which is the exact result [5]. The most obvious solution is given by the constant-density configuration $\rho = \rho_0$ for any value of the statistical parameter λ. However, there exists one non-trivial solution to equation (15), given by

$$\rho(\varphi) = \frac{N}{2\pi} \frac{\sqrt{a^2 - 1}}{a + \cos n\varphi} \tag{17}$$

where a is an arbitrary positive parameter, $a > 1$, and n is an integer given by

$$n = \frac{\lambda N}{\lambda - 1}. \tag{18}$$

Indeed, by using

$$\int \frac{d\varphi}{a + \cos n\varphi} = \frac{2\pi}{\sqrt{a^2 - 1}} \tag{19a}$$

$$\fint d\varphi' \ \cot \frac{\varphi - \varphi'}{2} \frac{1}{a + \cos n\varphi'} = -\frac{2\pi}{\sqrt{a^2 - 1}} \frac{\sin n\varphi}{a + \cos n\varphi} \tag{19b}$$

we can easily recover the solution (17) and the constraint (18). The solution (17) exists only for special values of the statistical parameter, given by (18). This constraint is a consequence of the periodicity condition

$$\rho(\varphi) = \rho(\varphi + 2\pi). \tag{20}$$

It represents some kind of stationary waves around the constant condensed state $\rho_0 = N/2\pi$.

Let us now find an interesting stationary hole-like excitation of particles in the Sutherland model, which can also be reached by the Bogomol'nyi saturation. Using the identity

$$\mathcal{P} \cot \frac{\varphi - \varphi'}{2} \mathcal{P} \cot \frac{\varphi - \varphi''}{2} + \mathcal{P} \cot \frac{\varphi' - \varphi}{2} \mathcal{P} \cot \frac{\varphi' - \varphi''}{2} + \mathcal{P} \cot \frac{\varphi'' - \varphi'}{2} \mathcal{P} \cot \frac{\varphi'' - \varphi}{2}$$

$$= 4\pi^2 \delta(\varphi - \varphi')\delta(\varphi - \varphi'') - 1 \tag{21}$$

we can rewrite the collective-field potential V_{eff} as

$$V_{\text{eff}} = \frac{1}{2} \int d\varphi \, \rho(\varphi) \left(\frac{\lambda - 1}{2} \frac{\partial_\varphi \rho(\varphi)}{\rho(\varphi)} + \frac{\lambda}{2} \!\!\!\fint d\varphi' \cot \frac{\varphi - \varphi'}{2} \rho(\varphi') + \frac{c}{2} \cot \frac{\varphi}{2} \right)^2 + \frac{c\lambda}{8} N^2$$

$$+ \frac{c - \lambda c - c^2}{8} \int d\varphi \, \rho(\varphi) \cot^2 \frac{\varphi}{2} - \frac{\lambda - 1}{8} cN - \frac{c\lambda\pi^2}{2} \rho^2(0)$$

$$+ \frac{c\lambda}{8} \left(\int d\varphi \, \rho(\varphi) \cot \frac{\varphi}{2} \right)^2. \tag{22}$$

For the symmetric configuration $\rho(\varphi) = \rho(-\varphi)$, representing a hole located at the origin, $\rho(0) = 0$, and for the particular value of the constant c given by

$$c = 1 - \lambda \tag{23}$$

the Bogomol'nyi limit appears. The contribution of the squared term in V_{eff} vanishes and the corresponding configuration satisfies the enlarged Bogomol'nyi equation

$$\frac{\lambda - 1}{2} \frac{\partial_\varphi \rho(\varphi)}{\rho(\varphi)} + \frac{\lambda}{2} \!\!\!\fint d\varphi' \cot \frac{\varphi - \varphi'}{2} \rho(\varphi') + \frac{1 - \lambda}{2} \cot \frac{\varphi}{2} = 0. \tag{24}$$

The new, singular term in equation (24) is to compensate for the singularity produced by $\partial_\varphi \ln \rho(\varphi)$ at the origin, $\varphi = 0$. Equation (24) can be solved by a rational ansatz, and the normalized, static solution is of the form

$$\rho(\varphi) = \frac{a \sin^2(\varphi/2)}{b^2 + \sin^2(\varphi/2)} \tag{25}$$

where the constants a and b satisfy the constraint

$$-\frac{2\lambda}{\lambda - 1} \frac{ab\pi}{\sqrt{1 + b^2}} = 1. \tag{26}$$

From equation (26) and the normalization condition

$$\int d\varphi \, \rho(\varphi) = N = 2\pi a \left(1 - \frac{b}{\sqrt{1 + b^2}} \right) \tag{27}$$

it follows that

$$a = \frac{N}{2\pi} + \frac{1 - \lambda}{2\pi\lambda} \qquad \text{and} \qquad b^2 = \frac{1}{(N\lambda/(1 - \lambda) + 1)^2 - 1}. \tag{28}$$

Since the collective-field density is positive, a and b are necessarily positive parameters and therefore it follows from relation (26) that $\lambda < 1$. The corresponding energy is given by

$$E = E_0 + \tfrac{1}{8} N(1 - \lambda)(\lambda N - \lambda + 1). \tag{29}$$

We are now going to show that there exists a multi-vortex solution to equation (24). For this purpose, we must rearrange the effective potential V_{eff} as follows:

$$V_{\text{eff}} = \frac{1}{2} \int d\varphi \, \rho(\varphi) \left(\frac{\lambda - 1}{2} \frac{\partial_\varphi \rho(\varphi)}{\rho(\varphi)} + \frac{\lambda}{2} \fint d\varphi' \, \cot \frac{\varphi - \varphi'}{2} \rho(\varphi') + c \cot \frac{n\varphi}{2} \right)^2$$

$$- c \int d\varphi \, \rho(\varphi) \cot \frac{n\varphi}{2} \left(\frac{\lambda - 1}{2} \frac{\partial_\varphi \rho(\varphi)}{\rho(\varphi)} + \frac{\lambda}{2} \fint d\varphi' \, \cot \frac{\varphi - \varphi'}{2} \rho(\varphi') \right)$$

$$- \frac{c^2}{2} \int d\varphi \, \rho(\varphi) \cot^2 \frac{n\varphi}{2}. \tag{30}$$

In order to get the Bogomol'nyi form, we should show that all terms in V_{eff}, except the first one, transform into an irrelevant constant functional. All $\int d\varphi \, \rho(\varphi) \cot^2(n\varphi/2)$ terms disappear if the strength c of the cotangent regulator term is given by

$$c = \frac{1 - \lambda}{2} n. \tag{31}$$

Using the summation formula [11]

$$n \cot \frac{n\varphi}{2} = \sum_{k=0}^{n-1} \cot \left(\frac{\varphi}{2} + \frac{k\pi}{n} \right) \tag{32}$$

and the principal-value identity (21), we can recast the final term in V_{eff} as

$$n \frac{\lambda(\lambda - 1)}{4} \int d\varphi \, d\varphi' \, \rho(\varphi)\rho(\varphi') \cot \frac{\varphi - \varphi'}{2} \cot \frac{n\varphi}{2}$$

$$= \frac{\lambda(\lambda - 1)}{4} \left[-n \frac{N^2}{2} + 2\pi^2 \sum_{k=0}^{n-1} \rho^2 \left(\frac{2k\pi}{n} \right) \right]. \tag{33}$$

Here we have assumed that $\rho(\varphi)$ is an even function in φ. Assuming further that $\rho(\varphi)$ describes the n-vortex-like configuration, with equidistant vanishing points at $\varphi_k = 2k\pi/n$, our V_{eff} functional finally reduces to

$$V_{\text{eff}} = \frac{(\lambda - 1)^2}{8} N n^2 - \frac{\lambda(\lambda - 1)}{8} N^2 n + \frac{1}{2} \int d\varphi \, \rho(\varphi) \left(\frac{\lambda - 1}{2} \frac{\partial_\varphi \rho(\varphi)}{\rho(\varphi)} \right.$$

$$\left. + \frac{\lambda}{2} \fint d\varphi' \, \cot \frac{\varphi - \varphi'}{2} \rho(\varphi') + n \frac{1 - \lambda}{2} \cot \frac{n\varphi}{2} \right)^2. \tag{34}$$

Thus we have achieved our goal and the minimal value of V_{eff} is given by the Bogomol'nyi saturation:

$$\frac{\lambda - 1}{2} \frac{\partial_\varphi \rho(\varphi)}{\rho(\varphi)} + \frac{\lambda}{2} \fint d\varphi' \, \cot \frac{\varphi - \varphi'}{2} \rho(\varphi') + n \frac{1 - \lambda}{2} \cot \frac{n\varphi}{2} = 0. \tag{35}$$

The corresponding energy is given by

$$E_n = E_0 + \frac{(\lambda - 1)^2}{8} N n^2 - \frac{\lambda(\lambda - 1)}{8} N^2 n. \tag{36}$$

For $n = 1$, the energy is equal to that found in (29). The enlarged Bogomol'nyi equation can again be solved by a rational ansatz

$$\rho(\varphi) = \frac{p \sin^2(n\varphi/2)}{q^2 + \sin^2(n\varphi/2)}. \tag{37}$$

Contour integration gives

$$\fint_{-\pi}^{\pi} d\varphi' \, \cot \frac{\varphi - \varphi'}{2} \rho(\varphi') = -\frac{pq\pi}{\sqrt{1 + q^2}} \frac{\sin n\varphi}{q^2 + \sin^2(n\varphi/2)}. \tag{38}$$

Substituting equation (38) into (35), we obtain the following condition for the positive parameters p and q:

$$-\frac{2\lambda}{\lambda - 1} \frac{pq\pi}{\sqrt{1 + q^2}} = n. \tag{39}$$

From relation (39) and the normalization condition equivalent to (27) we find that the parameters p and q read

$$p = \frac{1}{2\pi} \left(N + n \frac{1 - \lambda}{\lambda} \right) \tag{40a}$$

$$q^2 = \frac{1}{((\lambda/(1 - \lambda))(N/n) + 1)^2 - 1}. \tag{40b}$$

It is evident that the constraint (39) implies $\lambda < 1$.

4. Quantum corrections

Let us now turn our attention to the next-to-leading-order terms in the collective Hamiltonian (10). We are going to study the effect of the small density fluctuations around the hole-like configuration:

$$\rho(\varphi) = \rho_0(\varphi) + \eta(\varphi). \tag{41}$$

Introducing the operators

$$A(\varphi) = \partial_\varphi \pi(\varphi) + i \left[\frac{\lambda - 1}{2} \partial_\varphi \left(\frac{\eta(\varphi)}{\rho_0(\varphi)} \right) + \frac{\lambda}{2} \fint d\varphi' \, \cot \frac{\varphi - \varphi'}{2} \eta(\varphi') - \frac{\lambda - 1}{2} \cot \frac{\varphi}{2} \right] \tag{42a}$$

$$A^\dagger(\varphi) = \partial_\varphi \pi(\varphi) - i \left[\frac{\lambda - 1}{2} \partial_\varphi \left(\frac{\eta(\varphi)}{\rho_0(\varphi)} \right) + \frac{\lambda}{2} \fint d\varphi' \, \cot \frac{\varphi - \varphi'}{2} \eta(\varphi') - \frac{\lambda - 1}{2} \cot \frac{\varphi}{2} \right] \tag{42b}$$

with the c-number commutator

$$[A(\varphi), A^\dagger(\varphi')] = -(\lambda - 1)\partial_\varphi \partial_{\varphi'} \frac{\delta(\varphi - \varphi')}{\rho_0(\varphi)} + \lambda \partial_\varphi \cot \frac{\varphi - \varphi'}{2} \tag{43}$$

the collective Hamiltonian can be written up to the quadratic terms in η and π as

$$H = E_0 + \frac{N}{8}(1 - \lambda)(\lambda N - \lambda + 1) + \frac{1}{2} \int d\varphi \, \rho_0(\varphi) A^\dagger(\varphi) A(\varphi). \tag{44}$$

The divergent terms disappear, as can be easily checked using the commutator (43). The collective Hamiltonian is semi-definite and there exists the collective-field wavefunctional $\Phi(\eta)$ such that

$$A(\varphi)\Phi(\eta) = 0. \tag{45}$$

For this wavefunctional the correction due to fluctuations vanishes. Solving equation (45), we easily get

$$\Phi(\eta) = \exp\left\{\frac{\lambda-1}{4}\int d\varphi\,\frac{\eta^2(\varphi)}{\rho_0(\varphi)} + \frac{\lambda}{4}\int d\varphi\,d\varphi'\,\eta(\varphi)\ln\sin^2\frac{\varphi-\varphi'}{2}\eta(\varphi)\right.$$

$$\left. -\frac{\lambda-1}{2}\int d\varphi\,\ln\sin^2\frac{\varphi}{2}\eta(\varphi)\right\}. \tag{46}$$

From this result we can reconstruct the Schrödinger wavefunction $\Psi(\varphi_1, \ldots, \varphi_N)$ for the N-particle system, which corresponds to the one-hole configuration. It is given by

$$\Psi(\eta) = \Delta^\lambda J^{1/2}\Phi(\eta). \tag{47}$$

Here, the Δ prefactor is present owing to the extraction (3). The Jacobian of the transformation from φ_i into $\rho(\varphi)$ rescales the wavefunctional by the $J^{1/2}$ factor. By expanding the Jacobian to the quadratic terms in η and using the relations (46), (47) and the Bogomol'ny equation for ρ_0 (24), we are left with

$$\Psi(\eta) = \Delta^\lambda \exp\left[\frac{1-\lambda}{2}\int d\varphi\,\ln\sin^2\frac{\varphi}{2}\eta(\varphi)\right]. \tag{48}$$

If we substitute equation (6) in (48), we obtain the wavefunction for N particles:

$$\psi(\varphi_1, \ldots, \varphi_N) = \Delta^\lambda \prod_{i=1}^{N}\sin^{1-\lambda}\frac{\varphi_i}{2}. \tag{49}$$

It can easily be checked that this wavefunction does indeed describe the configuration with known energy (29), provided that $\rho(\varphi)$ satisfies

$$\int d\varphi\,\cot\frac{\varphi}{2}\rho(\varphi) = 0 \qquad \text{i.e.} \quad \sum_{i=1}^{N}\cot\frac{\varphi_i}{2} = 0. \tag{50}$$

The wavefunctional for the n-hole-like configuration can be formed along similar lines, explicitly given for the one-hole case. It reads

$$\psi_n(\varphi_1, \ldots, \varphi_N) = \Delta^\lambda \prod_{i=1}^{N}\sin^{n(1-\lambda)}\frac{n\varphi_i}{2}. \tag{51}$$

It can be shown that this wavefunction is indeed the eigenfunction of the N-particle Hamiltonian (1) with known energy (36), provided that $\rho(\varphi)$ satisfies

$$\int d\varphi\,\cot\frac{n\varphi}{2}\rho(\varphi) = 0 \qquad \text{i.e.} \quad \sum_{i=1}^{N}\cot\frac{n\varphi_i}{2} = 0. \tag{52}$$

Let as briefly comment on the corresponding quantum corrections in the case of the Calogero model. Owing to the Bogomol'nyi form we can in the same way show the stability of the static solitons [3] against first-order quantum corrections.

5. Summary

We have found two main results. The first result is that in the collective-field formulation of the Sutherland model there exist multi-vortex static configurations which could be reached by the Bogomol'nyi saturation. Since we know that in the Calogero model there exists a moving soliton [2, 3], it is of interest to look for the existence of moving-multi-vortex solutions.

The second result is that the energies of these configurations are not affected by the next-to-leading-order corrections stemming from the quantum collective-field fluctuations. This is what happens in the supersymmetric theory where the Bogomol'nyi bound does not receive quantum corrections [12]. Therefore we conclude that there must exist a supersymmetric extension of the Calogero–Sutherland model in the collective-field formulation. For $\lambda = 1$, this has already been done in [13].

Acknowledgments

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